ESTIMATION OF A STOCHASTIC VOLATILITY AND JUMPS MODEL USING GENERALIZED METHOD OF MOMENTS WITH ORDINARY MOMENT CONDITIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ŞEREF KUTAY YAKUT

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
FINANCIAL MATHEMATICS

APRIL 2024

Approval of the thesis:

ESTIMATION OF A STOCHASTIC VOLATILITY AND JUMPS MODEL USING GENERALIZED METHOD OF MOMENTS WITH ORDINARY MOMENT CONDITIONS

submitted by **ŞEREF KUTAY YAKUT** in partial fulfillment of the requirements for the degree of **Master of Science in Financial Mathematics Department, Middle East Technical University** by,

Prof. Dr. A. Sevtap Kestel ______________
Dean, Graduate School of **Applied Mathematics**

Prof. Dr. A. Sevtap Kestel ______________
Head of Department, **Financial Mathematics**

Prof. Dr. Ali Devin Sezer ______________
Supervisor, **Financial Mathematics, METU**

Examining Committee Members:

Prof. Dr. Özge Sezgin Alp
Accounting and Finance Management, Başkent University ______________

Prof. Dr. Ali Devin Sezer
Financial Mathematics, METU ______________

Assist. Prof. Dr. Büşra Temoçin
Actuarial Sciences, METU ______________

Date: ______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ŞEREF KUTAY YAKUT

Signature :

ABSTRACT

ESTIMATION OF A STOCHASTIC VOLATILITY AND JUMPS MODEL USING GENERALIZED METHOD OF MOMENTS WITH ORDINARY MOMENT CONDITIONS

Yakut, Şeref Kutay
M.S., Department of Financial Mathematics
Supervisor : Prof. Dr. Ali Devin Sezer

April 2024, 34 pages

One of the first works estimating jump risk premium in financial markets is the seminal work of Jun Pan published in 2002. In this work Pan uses the generalized method of moments (GMM) to estimate the parameters of a stochastic volatility price model with jumps from index and option price data. In the implementation of GMM, Pan uses a set of optimal moment conditions. In this thesis, we simulate the stochastic model used in Pan's work and apply the GMM estimation algorithm using ordinary moment conditions on simulated data. The estimation results suggest that the ordinary moment conditions are not very sensitive to model parameters and as a result the estimation algorithm quickly converges to a point around the initial parameter estimate. We applied the same algorithm to a stock price and a call option quoted on Borsa İstanbul and observed a similar performance.

Keywords: Jump-risk premium, option prices, stochastic differential equations, Borsa Istanbul

ÖZ

SIÇRAMALI VE STOKASTİK VOLATİLİTELİ BİR MODELİN SADE MOMENT KOŞULLU GENELLEŞTİRİLMİŞ MOMENTLER YÖNTEMİ İLE TAHMİNİ

Yakut, Şeref Kutay
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Ali Devin Sezer

Nisan 2024, 34 sayfa

Finansal marketlerde sıçrama riski priminin hesaplanmasıyla ilgili ilk çalışmalardan biri Jun Pan'in 2002 yılında yayınladığı yazısıdır. Pan bu çalışmasında sıçramaya izin veren bir stokastik volatilite modelindeki stokastik diferansiyel denklemin parametrelerini genelleştirilmiş momentler yöntemiyle (generalized method of moments, GMM) endeks ve opsiyon fiyat verisi kullanarak tahmin etmiştir. GMM uygulamasında Pan optimize edilmiş moment koşulları kullanmıştır. Bu çalışmamızın amacı, Pan'in makalesinde kullandığı modeli simüle etmek ve simüle edilen veri üzerinde tahmin çalışmasını sade moment koşulları kullanarak uygulamaktır. Tahmin algoritması simüle edilmiş verilere uygulandığında, sade moment koşullarının modelin parametrelerine çok duyarlı olmadığı ve bu sebeple algoritmanın sonucunun önemli ölçüde algoritmaya verilen başlangıç parametreleri tarafından belirlendiği gözlemlenmiştir. Aynı algoritma Borsa İstanbul'da işlem gören bir hisse senedine ve bu hisse senedi üzerine yazılı bir alım opsiyonuna da uygulanmıştır ve benzer sonuçlar elde edilmiştir.

Anahtar Kelimeler: Sıçrama Riski Primi,Opsiyon Fiyatlandrması, Stokastik Diferansiyel Denklemler, Borsa İstanbul

To My Mom & My Dad

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my advisor Prof. Dr. Ali Devin Sezer for his guidance and help in my thesis work.

I also would like to thank my jury members, Prof. Dr. Özge Sezgin Alp, and Assist. Prof. Dr. Büşra Temoçin for their valuable comments and discussions about this work.

I believe that one does not achieve the greatest only by themselves, they always need the support from their loved ones. I would like to give my deepest gratitude to my mom Emine, my dad Hüseyin, my grandmother Saniye, and my aunts İlknur and Nilgün. I could not complete this work without their endless love, support and encouragement. They are the main motivation for me to keep going.

I also would like to thank my dearest friends İlkyaz, Göktuğ, Aybüke, Barış, Ferhat, Azime, Eren and Elif for being with me to pass every bump on the way together.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Stochastic differential equations (SDE) are generalizations of ordinary differential equations (ODE) where the right side of the equation is allowed to have stochastic integral terms driven by stochastic processes such as the Brownian motion or more generally Levy processes. Starting with the seminal work [6] of Black and Scholes (BS) the SDE based models gained a central role in option pricing. The initial BS model assumes that stock prices are driven by a standard Brownian motion with constant volatility, i.e., the asset price is assumed to satisfy an SDE of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\sigma > 0$ is a constant and W is a standard Brownian motion. A variety of works presented to develop the Black Scholes [6] formula with implementing jump and volatility processes such as Bakshi [3] and Chernov [8]. An important work concerning jump processes is [5] by Bates. He presented evidence that the distribution of S&P 500's option prices can be modeled accurately using stochastic volatility process with jumps for small volatility shocks. Various treatments applied to Bates model such as a fast numerical solution using a Bermudian approximation by Ballestra [4] and a linear complementarity problem formulation by Toivanen [21]. Duffie, Pan and Singleton [10] presented an analytical framework for option pricing for generalized affine jump diffusions using characteristic functions and Fourier transforms. A similar characteristic function treatment done by Deng [9] for a two-factor stochastic volatility model. In addition to being the main tools in option pricing, these models can be used to understand the structure of financial markets. A natural question in this regard is the following: assuming that prices in a given market exhibits random jumps, how is this jump-risk priced in the market? To the best of our knowledge, the first work that asks and systematically treats this question is the seminal work [16] by Jun Pan. In [16], Pan fits an SDE model that allows stochastic volatility and jumps to the S&P 500 data to compute a risk-premium for jump risk. The goal of this thesis is to simulate the model in this paper and try to fit it to Borsa Istanbul data using a simplified version of the estimation procedure in [16].

The specific SDE model that [16] uses for the stock price and volatility processes is given in Chapter 2. In this introduction we will give a summary of Pan's [16] approach in estimating jump risk premium and explain the content of our thesis work. Pan's uses a model for (S, V) that is similar to the Bates model with denoting S as the price process and V as the stochastic volatility process. In addition to (S, V), he assumes that the dividend process q and the interest rate processes r to be stochastic which are to be estimated using market data. As in the classical Black Scholes model, risk premiums, including the jump-risk premium, are encoded in the difference between the drifts of the price process S under the actual probability measure governing (S, V) and the pricing risk neutral measure. For S, Pan [16] uses the S&P 500 index itself; however for stochastic volatility V, he uses the implied volatility which is obtained from C, the price of a call option, since volatility itself can not directly observed in the market. For C Pan uses a call option on S with strike K that is closest to S and a maturity that is as near as possible to today's date in the market. Note that to get V from C one needs the model parameters, which are unknown. So in each iteration of the estimation process [16] uses the parameter values in the current estimation step; we comment on this further below.

As in the Bates model in [5], the model assumptions allow a direct calculation of the characteristic function of (S, V). This is the main computational tool both in the computation of option prices and model estimation. The characterstic functions used in [16] are derived in [10]. In Section 2.4 we rederive the characteristic function of (S, V) assuming q and r to be constant.

For estimating the parameters of the stochastic volatility model, there can be found a variety of different methods in the literature. Friedman and Harris [12] presented a likely-hood estimation approach using recursive numerical integration and Sandmann

[19] showed Monte Carlo likely-hood method of estimating models with comparing Monte Carlo Markov Chain approach. Multiple adaptations for method of moments method is also presented, such as Duffie [11] and Bolko [7]. Andersen and Sorensen [2] introduced generalized method of moments to stochastic models by further offering how to select moments and the weighting matrix to get desirable results in small samples. As the estimation procedure Pan uses GMM. The GMM is applicable because the availability of the characteristic function of (S, V) enables an explicit calculation of the joint moments. However, there is an issue that needs to be handled before GMM estimation: V itself is not directly observable. To overcome this problem, Pan uses the following approach: let ϑ_n be the sequence of parameter estimates generated by the estimation process, let $V_t^{\vartheta_n}$ be the volatility implied by (S_t, V_t) assuming that the actual model parameters equal ϑ_n; then in the $(n+1)^{st}$ iteration of GMM (S, V^{ϑ_n}) is used as the underlying data; Pan calls this procedure Implied State GMM (IS-GMM) in [16]. We will give further comment about IS-GMM in Chapter 3.

Our thesis work consists of the following: in Section 2.4 we give a detailed derivation of the characteristic function of (S, V) assuming r and q to be constant. In Chapter 4 we simulate (S, V) and the compute C for the simulated data. We then apply the IS-GMM algorithm of Pan to the simulated data. In our implementation of the GMM we directly use the ordinary moment conditions rather than the optimal moment conditions used in [16]. We then present a first attempt at applying this approach to computing jump risk premium to Borsa Istanbul data. BIST30 Index consists of 30 of the major stocks traded on Borsa Istanbul. We use as S one of the components of the BIST30 index: TUPRS, Turkiye Petrol Rafinerileri AS. As of January 2024, TUPRS constitutes around 7% of the market capitalization of BIST30. In order to ease our calculations, we also assumed that the interest rate r to be the Central Bank of Turkey's one-week repo auction rate ($r = 0.45$) as in February 2024. For our analysis, we chose a call option with the underlying TUPRS between dates of February 1 and March 15 of 2024. We comment on the results of the application of IS-GMM to simulated data and to TUPRS in Chapter 4. Conclusion (Chapter 5) comments on possible future work.

CHAPTER 2

MODEL

2.1 Data Generating Process

Throughout this work, the random variables are defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the filtration $\{\mathscr{F}_t\}$, which satisfies the usual conditions (see, [17, Chapter 1]). In this section, we follow the adaptation of the Bates [5] model, made by Pan [16]. Heston's model contains three elements of uncertainty to the underlying price dynamic: a diffusive return shock, volatility shock and jump risk. Before stating the data generating process, we introduce the jump dynamics in the model.

2.1.1 Jump Dynamics

In the model, the price process follows a pure jump process. A pure jump process is a purely discontinuous stochastic process such as a Poisson Process. Jumps occur with a Poisson Counter N_t with a state-dependent stochastic intensity process $\{\lambda V_t : t > 0\}$ with $\lambda > 0$. If a jump occurs at time $t = \tau$, the stock price jumps with from $S(\tau_-)$ to $S(\tau_-)\text{exp}(U_\tau^s)$ where U_i^s is normally distributed with $U_i^s \sim \text{N}(\mu_j, \sigma_j)$. This specification creates a jump size of $(\text{exp}(U_\tau^s) - 1)$. Pan defines the jump dynamics as a Compound Poisson process with the help of a definition in [20, Section 11.3] as follows:

$$Z_t = \sum_{i=1}^{N_t} \text{exp}(U_i^s) - 1 \tag{2.1}$$

where N_t is independent of $(\exp(U_i^s) - 1)$ with $\mu = \mathbb{E}[\exp(U^s) - 1] = \exp(\mu_j + \sigma_j^2/2) - 1$ as mean relative jump size. Using μ, one can define compensated compound process as follows:

Theorem 2.1.1. *Let Z_t be a compound Poisson process defined as in* (2.1)*. Then the compensated Poisson process*

$$Z_t - \lambda V_t \mu t$$

is a martingale.

Proof. See [20, Theorem 11.3.1]. □

2.1.2 Data Generation

The data generating process introduced by Pan [16] is as follows:

$$\begin{aligned} dS_t &= [r_t - q_t + \eta^s V_t + \lambda V_t(\mu - \mu^*)]S_t dt + \sqrt{V_t} S_t dW_t^{(1)} + dZ_t - \mu S_t \lambda V_t dt \\ dV_t &= \kappa_v[\bar{v} - V_t]dt + \sigma_v \sqrt{V_t}\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right) \end{aligned} \tag{2.2}$$

where $W = [W^{(1)}, W^{(2)}]^{\mathrm{T}}$ is a standard Brownian motion, Z_t is a compound Poisson process, independent of W, as described in Section 2.1.1 with both being adopted to the filtration $\{\mathscr{F}_t\}$.

Focusing on the equity risk premia (or drift) of the price process, stock pays out dividend yield q_t, appreciates r_t as constant interest rate and two risk-premium components; $\eta^s V_t$ and $\lambda V_t(\mu - \mu^*)$. Risk-premium for Brownian return risks are treated similarly to the risk-return trade-off of the Capital Asset Pricing Model (CAPM). It is parameterized by $\eta^s V_t$ where η^s is a constant. Risk-premium for volatility risks are not as clear as return risks, since volatility itself is not an asset to be traded. However, Pan parameterized it with parameter $\eta^v V_t$ by mentioning volatility of volatility may reflect an additional premium in options. This additional parameter will be introduced to model in (2.6).

Jump risks are priced on the market by allowing μ^*, the risk-neutral jump size, to be different from the data-generating counterpart μ. The jump-timing risk can be measured similarly by allowing risk-neutral jump time parameter λ^* to be different from the data-generating counterpart λ. In his work, Pan focuses on the jump size risk premium implicit in options; therefore we will ignore jump timing risk premium by assuming $\lambda^* = \lambda$. While calculating risk-neutral measure, this specification contributes to the pure-jump process Z_t to be a martingale under risk-neutral measure $\mathbb{Q}$.

The volatility process defined as an one-factor square root process, where κ_v is mean reversion rate, $\bar{v}$ is constant long-run mean and σ_v is the volatility coefficient of volatility. Volatility and price processes are correlated by ρ as introduced in [15].

2.2 Risk-Neutral Measure

In contrast of Black Scholes setting, the model contains additional parameters of uncertainty such as jump-risks which makes the market incomplete. In order to eliminate arbitrage opportunities, we follow a change of measure technique given by Shreve as in [20, Section 11.6.3].

Proposition 2.2.1. *Let us define a Brownian motion W_t and compound Poisson process Z_t as in Section 2.1.1 with density $f(y)$ on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let λ^* be a positive number, let $\tilde{f}(y)$ be another density function with property $\tilde{f}(y) = 0$ whenever $f(y) = 0$ and ζ_t be an adapted process. Define π_t as follows:*

$$\pi_t^1 = exp\left(-\int_0^t \zeta_u dW_u - \frac{1}{2}\int_0^t \zeta_u^2 du\right) \tag{2.3}$$

$$\pi_t^2 = e^{(\lambda-\lambda^*)t} \prod_{i=1}^{N(t)} \frac{\lambda^* \tilde{f}(Y_i)}{\lambda f(Y_i)} \tag{2.4}$$

$$\pi_t = \pi_t^1 \pi_t^2 \tag{2.5}$$

The process π_t is a martingale.

Proof. See [20, Lemma 11.6.8]. □

The model contains two independent Brownian Motions. To price the additional risk factors, we need to specify the adopted ζ_t for each Brownian Motion. In [16, Appendix A], Pan defines ζ_t as follows:

$$\zeta_t^{(1)} = \eta^s \sqrt{V_t} \qquad \zeta_t^{(2)} = -\frac{1}{\sqrt{1-\rho^2}} \left(\rho \eta^s + \frac{\eta^v}{\sigma_v} \right) \sqrt{V_t} \tag{2.6}$$

where $\zeta_t^{(1)}$ and $\zeta_t^{(2)}$ are adopted processes for $W_t^{(1)}$ and $W_t^{(2)}$, respectively. If we plug (2.6) into (2.3), we will obtain the following expressions:

$$\pi_t^1(1) = \exp\left(-\int_0^t \eta^s \sqrt{V_u} dW_u^{(1)} - \frac{1}{2} \int_0^t (\eta^s)^2 V_u du \right) \tag{2.7}$$

$$\begin{aligned}\pi_t^1(2) = \exp\Bigg(&-\int_0^t \frac{-1}{\sqrt{1-\rho^2}} \left(\rho \eta^s + \frac{\eta^v}{\sigma_v} \right) \sqrt{V_u} dW_u^{(2)} \\ &- \frac{1}{2} \int_0^t \frac{1}{1-\rho^2} \left(\rho \eta^s + \frac{\eta^v}{\sigma_v} \right)^2 V_u du \Bigg)\end{aligned} \tag{2.8}$$

where $\pi_t^1(1)$ and $\pi_t^1(2)$ are for $W_t^{(1)}$ and $W_t^{(2)}$, respectively.

Pan assumes the following approach for jump process: First, he assumes $\lambda^* = \lambda$ to keep jump-time intensity same as data generating process. Next, he assumes jump-sizes are distributed log-normally with $\mathbb{Q}$-mean μ_j^* and $\mathbb{Q}$-variance σ_j^2. Deploying our setting in (2.4) gives us the following:

$$\begin{aligned}\pi_t^2 &= \prod_{i=1}^{N(t)} \frac{\tilde{f}(Y_i)}{f(Y_i)} \\ &= \prod_{i=1}^{N(t)} \frac{\frac{1}{y_i \sigma_j \sqrt{2\pi}} \exp\left(-\frac{(\ln y_i - \mu_j^*)^2}{2\sigma_j^2} \right)}{\frac{1}{y_i \sigma_j \sqrt{2\pi}} \exp\left(-\frac{(\ln y_i - \mu_j)^2}{2\sigma_j^2} \right)} \\ &= \prod_{i=1}^{N(t)} \exp\left(\frac{(\mu_j^* - \mu_j)(2\ln y_i - \mu_j - \mu_j^*)}{2\sigma_j^2} \right)\end{aligned} \tag{2.9}$$

Under risk neutral measure $\mathbb{Q}$, the jump size risk premium in (2.2) creates a compensating effect for the compound Poisson process.

Now, define $\pi_t = [\pi_t(1), \pi_t(2)]^\top$ where $\pi_t(i) = \pi_t^1(i) * \pi_t^2$ with $i = 1, 2$ as in (2.7) and (2.8), respectively. To find a equivalent martingale measure $\mathbb{Q}$, fix a maturity date

T and define $\mathbb{Q} = \int_{\mathcal{A}} \pi_t d\mathbb{P}$ for all $\mathcal{A} \in \mathcal{F}$. Now, we give a theorem which connects data-generating measure $\mathbb{P}$ to risk-neutral measure $\mathbb{Q}$.

Theorem 2.2.1. *Under probability measure $\mathbb{Q}$, the process*

$$\widetilde{W}_t = W_t + \int_0^t \zeta_s ds \tag{2.10}$$

is Brownian Motion, Z_t is compound Poisson process with jump time intensity λ^* and independent, identically jump distributed jump sizes having density $\tilde{f}(Y_i)$, and the processes $\widetilde{W}_t$ and Z_t are independent.

Proof. See, [20, Theorem 11.6.9]. □

Using Theorem 2.2.1, one can find the Brownian Motion $\widetilde{W}_t$ under $\mathbb{Q}$ measure as follows:

$$\begin{aligned} d\widetilde{W}_t^{(1)} &= dW_t^{(1)} + \eta^s \sqrt{V_t} dt \\ d\widetilde{W}_t^{(2)} &= dW_t^{(2)} - \frac{1}{\sqrt{1-\rho^2}} \left(\rho \eta^s + \frac{\eta^v}{\sigma_v} \right) \sqrt{V_t} dt \end{aligned} \tag{2.11}$$

The risk-neutral price and volatility processes can be found by plugging (2.11) into (2.2).

$$\begin{aligned} dS_t &= [r_t - q_t] S_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^{(1)} + d\widetilde{Z}_t - \mu^* S_t \lambda V_t dt \\ dV_t &= [\kappa_v(\bar{v} - V_t) + \eta^v V_t] dt + \sigma_v \sqrt{V_t} \left(\rho d\widetilde{W}_t^{(1)} + \sqrt{1-\rho^2} d\widetilde{W}_t^{(2)} \right) \end{aligned} \tag{2.12}$$

where $\widetilde{W}_t = [\widetilde{W}_t^{(1)}, \widetilde{W}_t^{(2)}]^\top$ is standard Brownian Motion, $\widetilde{Z}_t$ is a pure jump process with jump arrival intensity $\{\lambda V_t : t \geq 0\}$. The jump-amplitudes U_i^s are normally distributed with mean μ_j^* and variance σ_j^2. The mean relative jump size is $\mu^* = \mathbb{E}^{\mathbb{Q}}[\exp(U^s) - 1] = \exp(\mu_j^* + \sigma_j^2) - 1$. As in the data generating measure, the last term $\mu^* S_t \lambda V_t dt$ is a compensator for jump process $\widetilde{Z}_t$. The volatility process under $\mathbb{Q}$ follows the same specification as in (2.2) except that the parameter $\eta^v V_t$ which captures volatility risk premium.

2.3 Option Pricing

The most influential work on option pricing done by Heston in [15]. In his work, he showed that risk-neutral probabilities in the option pricing formulas can be calculated by using Fourier inversion of a explicitly known conditional characteristic function of a stochastic volatility model. In this section, we will share an analytically tractable method for valuing a plain vanilla call option under S_t as in [10]. We denote the model parameters in (2.12) as:

$$\vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_j, \eta^s, \eta^v, \mu^*) \tag{2.13}$$

2.3.1 Linking Characteristic Function to Option Prices

The price of a call option at time-t can be denoted as C_t. Let us consider a future time T where C_T has a payoff $\max(0, e^{c.\ln S_T} - K)$ where S_T is a jump-diffusion process as in (2.12), $c \in \mathbb{C}$, and K is the exercise price of the option. The option is in the money when $\ln K \le c.\ln S_T$ with payoff $e^{c.\ln S_T} - K.e^{0.\ln S_T}$. In [10], Duffie et al. denoted $G_{a,b}(y)$ as the price of a security that pays $e^{a.X_T}$ at time T with $b.X_T \le y$. The implementation of this notation to the model gives the following:

$$C_t = G_{c,c}(\ln K) - KG_{0,c}(\ln K) \tag{2.14}$$

In [10], Duffie et al. observed that $G_{a,b}(y)$ can be treated as a measure since it is an increasing function. Therefore, they can compute the Fourier Transform of $\mathscr{G}_{a,b}$ of $G_{a,b}$ defined by

$$\mathscr{G}_{a,b}(z) = \int_{-\infty}^{+\infty} e^{izy} dG_{a,b}(y) \tag{2.15}$$

They computed the expected present value for option when it is well-defined for each given $(d, K, T) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+$,

$$\begin{aligned} C(c, K, T, \vartheta) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_u du\right) \max(0, e^{c\ln S_T} - K)\right] \\ &= G_{c,c}(\ln K; \ln S_0, T, \vartheta) - KG_{0,c}(\ln K; \ln S_0, T, \vartheta) \end{aligned} \tag{2.16}$$

where for $(x, T, a, b) \in D \times [0, \infty] \times \mathbb{R}^n \times \mathbb{R}^n$, $G_{a,b}(\cdot; \ln S_0, T, \vartheta) : \mathbb{R} \to \mathbb{R}$ is given by

$$G_{a,b}(y; \ln S_0, T, \vartheta) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_u du\right) e^{a\ln S_T} \mathbb{1}_{b\cdot \ln S_T \le y}\right] \tag{2.17}$$

Now, using (2.15),when well defined, the transform is given by

$$\begin{aligned}\mathscr{G}_{a,b}(v;\ln S_0,T,\vartheta) &= \int_{\mathbb{R}} e^{ivy} dG_{a,b}(y;\ln S_0,T,\vartheta) \\ &= \psi^{\vartheta}(a+ivb,\ln S_0,0,T)\end{aligned} \tag{2.18}$$

where ψ^{ϑ} is the conditional characteristic function of $\ln S_T$ with the following form

$$\psi^{\vartheta}(c,\ln S_t,t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_u du\right) e^{c\cdot\ln S_T}\middle|\mathscr{F}_t\right] \tag{2.19}$$

We share a proposition from [10] which extends the Lévy Inversion Formula to the transform.

Proposition 2.3.1. *(Transform Inversion) Suppose, for fixed $T \in [0,\infty]$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, that ϑ is well-behaved at $(a+ivb,T)$ for any $v \in \mathbb{R}$, and that*

$$\int_{\mathbb{R}} |\psi^{\vartheta}(a+ivb,x,0,T)|dv < \infty. \tag{2.20}$$

Then $G_{a,b}(\cdot;x,T,\vartheta)$ is well-defined by (2.17) *and given by*

$$\begin{aligned}G_{a,b}(y;lnS_0,T,\vartheta) =& \frac{\psi^{\vartheta}(a,lnS_0,0,T)}{2} \\ &- \frac{1}{\pi}\int_0^{\infty} \frac{Im[\psi^{\vartheta}(a+ivb,lnS_0,0,T)e^{-ivy}]}{v}dv\end{aligned} \tag{2.21}$$

Proof. See, [10, Appendix A] □

2.4 Characteristic Function

To facilitate call price analysis with stochastic model, one need to find the given option's moneyness probabilities on the underlying asset. However, finding the distribution of a stochastic model directly is a challenging endeavor. Characteristic functions gradually decrease the effort of calculating the distribution of a random variable by both being in a simple form as $\psi(u) = \mathbb{E}[e^{iuX_t}]$ and completely defining a random variable X_t's probability distribution. In this section, we derive a simplified version of the characteristic function of the risk-neutral S_t as presented in [16] using Ito's Formula.

2.4.1 Ito's Formula with Jumps

A crucial step for solving stochastic differential equations is to use Ito's Formula. The model contains two stochastic differential equations with one having jumps, therefore we need to work with generalized Ito's formula. In this section, we share the Ito's Formula for jump processes, then we state generalized Ito's Formula for two SDE's as described in [20, Section 11.4].

Let us start with an stochastic process X_t with the following decomposition

$$X_t = X_0 + R_t + I_t + J_t \tag{2.22}$$

In (2.22), X_0 is a non-random initial condition. The process R_t defined as a Riemann Integral

$$R_t = \int_0^t \theta_s ds \tag{2.23}$$

for some adopted θ_s. The process I_t in (2.22)

$$I_t = \int_0^t \Gamma_s dW_s \tag{2.24}$$

is an Ito Integral of an adapted Γ_s with respective to Brownian Motion. The continuous part of (2.22) denoted as X_t^C is

$$X_t^C = X_0 + R_t + I_t \tag{2.25}$$

with quadratic variation

$$[X_t^C, X_t^C] = \int_0^t \Gamma_s^2 ds \tag{2.26}$$

The jump part J_t in (2.22) is a right-continuous adapted pure jump process. We denote J_{t-} as the process just before jump occurs. Right continuity means that $J_t = \lim_{s\to t} J_s \ \forall t \geq 0$. Since X_0, R_t and I_t is continuous, the left continuous version of X_t is as follows:

$$X_{t-} = X_0 + R_t + I_t + J_{t-} \tag{2.27}$$

The jump size of X at time t can be given as

$$\Delta X_t = J_t - J_{t-} \tag{2.28}$$

Now, we can define the Ito's Integral as follows.

Definition 2.4.1 (Ito Process)**.** *Let X_t be a stochastic process as given in* (2.22) *and let Φ_t an adopted process. The stochastic integral of Φ with respect to X is given as*

$$\int_0^t \Phi_s dX_s = \int_0^t \Phi_s \theta_s ds + \int_0^t \Phi_s \Gamma_s dW_s + \sum_{0<s\leq t} \Phi_s \Delta J_s \tag{2.29}$$

In differential notion

$$\Phi_t dX_t = \Phi_t \theta_t dt + \Phi_t \Gamma_t dW_t + \Phi_t dJ_t \tag{2.30}$$

Using Ito's Process, we can define multidimensional Ito-Doeblin formula for jump processes which we later use to calculate the log-stock price lnS_t and characteristic function of the model.

Theorem 2.4.1 (Multidimensional Ito-Doeblin Formula)**.** *Let X_t^1 and X_t^2 be two jump processes defined as in* (2.22) *and let $f(t, x_1, x_2)$ be a function whose first and second order partial derivatives are defined and continuous. Then*

$$\begin{aligned}
&f(t, X_1(t), X_2(t)) \\
=&f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))ds \\
&+ \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^C(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^C(s) \\
&+ \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))dX_1^C(s)dX_1^C(s) \\
&+ \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))dX_2^C(s)dX_2^C(s) \\
&+ \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))dX_1^C(s)dX_2^C(s) \\
&+ \sum_{0<s\leq t} [f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))]
\end{aligned} \tag{2.31}$$

Proof. See [20, Theorem 11.5.4] □

2.4.2 Characteristic Function

Before working on the characteristic function, we need to find the log-normal stock price lnS_t. Let $f(x) = \text{ln}x$ with $f(x) \in C^2$ and S_t be the price process. Ito-Doeblin Formula for jumps gives us the following:

$$
\begin{aligned}
d(\ln S_t) &= \frac{1}{S_t}\Big([r_t - q_t - \mu^*\lambda V_t]S_t d_t + \sqrt{V_t}S_t d\widetilde{W}_t^{(1)}\Big) \\
&\quad - \frac{1}{2S_t^2}V_t S_t^2 d\widetilde{W}_t^{(1)} d\widetilde{W}_t^{(1)} + [(\ln S_t) - (\ln S_{t-})]dN_t \\
&= [r_t - q_t - \frac{V_t}{2} - \mu^*\lambda V_t]dt + \sqrt{V_t}S_t d\widetilde{W}_t^{(1)} + U^s dN_t
\end{aligned}
\tag{2.32}
$$

where $[d\widetilde{W}_t^{(1)}, d\widetilde{W}_t^{(1)}] = dt$ and $[(\ln S_t) - (\ln S_{t-})] = \ln(e^{U^s}S_t) - \ln(S_t) = U^s$

The characteristic function $\psi(c, V_t, T-t)$ can be found using $\ln S_t$ and V_t. For the remaining parts of this work, we will denote $\ln S_t$ as X_t for ease of notation. Under integrability conditions as in [10], let $\psi^\vartheta(c, v, T-t) = \exp(A(c, T-t) + B(c, T-t)v + cX_t)$ with boundary condition $\psi^\vartheta(c, v, 0) = e^{cX_T}$ at $t = T$. Partial derivatives for Multidimensional Ito's Formula are as follows:

$$
\begin{aligned}
&\frac{\partial \psi}{\partial t} = (A'(c, T-t) + B'(c, T-t))\psi, \quad \frac{\partial \psi}{\partial X_t} = c\psi, \quad \frac{\partial^2 \psi}{\partial X_t^2} = c^2\psi, \\
&\frac{\partial \psi}{\partial V_t} = B(c, T-t)\psi, \quad \frac{\partial^2 \psi}{\partial V_t^2} = B(c, T-t)^2\psi, \quad \frac{\partial^2 \psi}{\partial X_t V_t} = cB(c, T-t)\psi,
\end{aligned}
\tag{2.33}
$$

with quadratic variations $[dX_t, dX_t] = dt$, $[dV_t, dV_t] = \sigma_v^2 dt$ and $[dX_t, dV_t] = \sigma_v V_t \rho dt$. We further assume $\lambda = \lambda_0 + \lambda_1 V_t$ to keep the affinity structure. Using Theorem 2.4.1 gives,

$$
\begin{aligned}
\partial \psi =& (A'(c, T-t) + B'(c, T-t)V_t)\psi dt \\
&+ c\psi\big[(r_t - q_t - (\lambda_0 + \lambda_1 V_t)\mu^* - \frac{1}{2}V_t)dt + \sqrt{V_t}d\widetilde{W}_t^{(1)}\big] + \frac{1}{2}c^2\psi V_t dt \\
&+ B(c, T-t)\psi\big[(\kappa(\bar{v} - V_t) + \eta^v V_t)dt + \sigma_v\sqrt{V_t}(\rho d\widetilde{W}_t^{(1)} + \sqrt{1-\rho^2}d\widetilde{W}_t^{(2)})\big] \\
&+ \frac{1}{2}B^2(c, T-t)\psi\sigma_v^2 V_t dt + cB(c, T-t)\psi\sigma_v V_t \rho dt \\
&+ (\lambda_0 + \lambda_1 V_t)[\psi(c, X_t + U^s, V_t, T-t) - \psi(c, X_t, V_t, T-t)]dt
\end{aligned}
\tag{2.34}
$$

Martingale property implies that $\mathbb{E}^{\mathbb{Q}}[\partial\psi|\mathscr{F}_t] = 0$ a.s. Taking conditional expectation on the both sides of (2.34) gives:

$$
\begin{aligned}
0 =& (A'(c, T-t) + B'(c, T-t)V_t)\psi + c\psi[r_t - q_t - (\lambda_0 + \lambda_1 V_t)\mu^* - \frac{1}{2}V_t] \\
&+ \frac{1}{2}c\psi V_t^2 + B(c, T-t)\psi[\kappa(\bar{v} - V_t) + \eta^v V_t] + \frac{1}{2}B^2(c, T-t)\psi\sigma_v^2 V_t \\
&+ c\psi B(c, T-t)\sigma_v V_t \rho + (\lambda_0 + \lambda_1 V_t)[\mathbb{E}^{\mathbb{Q}}[\psi(c, X_t + U^s, V_t, T-t)] - \psi]
\end{aligned}
\tag{2.35}
$$

Using the fact that $\mathbb{E}^{\mathbb{Q}}[\psi(c, X_t + U^s, V_t, T-t)] = \psi \cdot \exp(\mu_s c + \frac{1}{2}\sigma_j^2 c^2)$ and dividing each side to ψ gives,

$$
\begin{aligned}
0 =& (A'(c, T-t) + B'(c, T-t)V_t) + c[r_t - q_t - (\lambda_0 + \lambda_1 V_t)\mu^* - \frac{1}{2}V_t] \\
&+ \frac{1}{2}cV_t^2 + B(c, T-t)[\kappa(\bar{v} - V_t) + \eta^v V_t] + \frac{1}{2}B^2(c, T-t)\sigma_v^2 V_t \\
&+ cB(c, T-t)\sigma_v V_t \rho + (\lambda_0 + \lambda_1 V_t)[\exp(\mu_j^* c + \frac{1}{2}\sigma_j^2 c^2) - 1]
\end{aligned}
\tag{2.36}
$$

We can divide (2.36) into 2 separate ODE's as follows:

$$
\begin{aligned}
0 = B'(c, T-t) &+ \frac{1}{2}\sigma_v^2 B^2(c, T-t) + B(c, T-t)(c\sigma_v\rho - \kappa + \eta^v) \\
&+ \left(\frac{c(c-1)}{2} + \lambda_1\left(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\right)\right)
\end{aligned}
\tag{2.37}
$$

$$
\begin{aligned}
0 = A'(c, T-t) &+ c(r_t - q_t) + B(c, T-t)\kappa\bar{v} \\
&+ \lambda_0\left(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\right)
\end{aligned}
\tag{2.38}
$$

Notice that (2.38) contains $B(c, T-t)$, therefore, once needs to solve (2.37) first. To simplicity, let $a = c(1-c) - 2\lambda_1(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*)$, and $b = c\sigma_v\rho - \kappa + \eta^v$. Rearranging the equation gives,

$$
\frac{dB(c, T-t)}{dt} = -\frac{1}{2}\sigma_v^2\left(B^2(c, T-t) + \frac{2b}{\sigma_v^2}B(c, T-t) - \frac{a}{\sigma_v^2}\right) \tag{2.39}
$$

$$
\frac{dB(c, T-t)}{B^2(c, T-t) + \frac{2b}{\sigma_v^2}B(c, T-t) - \frac{a}{\sigma_v^2}} = -\frac{1}{2}\sigma_v^2 dt \tag{2.40}
$$

Integrate both sides,

$$
\int_t^T \frac{1}{B^2(c, T-u) + \frac{2b}{\sigma_v^2}B(c, T-u) - \frac{a}{\sigma_v^2}} dB(c, T-u) = \int_t^T -\frac{1}{2}\sigma_v^2 du \tag{2.41}
$$

The denominator in left side of (2.41) is quadratic, we can find the roots using quadratic formula,

$$
\int_t^T \frac{1}{\left(B(c, T-u) + \frac{b+\gamma}{\sigma_v^2}\right)\left(B(c, T-u) - \frac{\gamma-b}{\sigma_v^2}\right)} dB(c, T-u) = \int_t^T -\frac{1}{2}\sigma_v^2 du \tag{2.42}
$$

with $\gamma^2 = b^2 + a\sigma_v^2$. The integral in the left hand side of (2.42) can be separated into two parts using method of partial fractions.

$$\frac{1}{\left(B(c,T-u)+\frac{b+\gamma}{\sigma_v^2}\right)\left(B(c,T-u)-\frac{\gamma-b}{\sigma_v^2}\right)} = \frac{\sigma_v^2}{2\gamma}\left(\frac{-1}{\left(B(c,T-u)+\frac{b+\gamma}{\sigma_v^2}\right)}+\frac{1}{\left(B(c,T-u)-\frac{\gamma-b}{\sigma_v^2}\right)}\right) \quad (2.43)$$

Rewriting LHS of (2.42) with RHS of (2.43) gives the following:

$$\frac{\sigma_v^2}{2\gamma}\left(\int_t^T \frac{-dB(c,T-u)}{\left(B(c,T-u)+\frac{b+\gamma}{\sigma_v^2}\right)}+\int_t^T \frac{dB(c,T-u)}{\left(B(c,T-u)-\frac{\gamma-b}{\sigma_v^2}\right)}\right) = \frac{-1}{2}\sigma_v^2\int_t^T du \quad (2.44)$$

The integrals can be found analytically,

$$\frac{-1}{\gamma}\left(\ln\left(B(c,0)+\frac{b+\gamma}{\sigma_v^2}\right)-\ln\left(B(c,T-t)+\frac{b+\gamma}{\sigma_v^2}\right)\right) + \frac{1}{\gamma}\left(\ln\left(B(c,0)+\frac{\gamma-b}{\sigma_v^2}\right)-\ln\left(B(c,T-t)+\frac{\gamma-b}{\sigma_v^2}\right)\right) = -(T-t) \quad (2.45)$$

Rearranging (2.45) and taking the exponents of both sides gives:

$$\frac{B(c,T-t)b\sigma_v^2 - B(c,T-t)\gamma\sigma_v^2 + b^2-\gamma^2}{B(c,T-t)b\sigma_v^2 + B(c,T-t)\gamma\sigma_v^2 + b^2-\gamma^2} = e^{-\gamma(T-t)} \quad (2.46)$$

Let us call $\tau = T - t$. We can leave $B(c,\tau)$ alone to obtain the solution:

$$B(c,\tau) = \frac{-a(1-e^{-\gamma\tau})}{2\gamma-(\gamma+b)(1-e^{-\gamma\tau})} \quad (2.47)$$

Now, we can find $A(c,\tau)$ using (2.47),

$$-A'(c,T-t) = c(r_t-q_t) + B(c,T-t)\kappa\bar{v} + \lambda_0\left(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\right) \quad (2.48)$$

Taking the integral with respect to t on both sides,

$$\int_t^T -dA(c,T-u) = \int_t^T c(r_t-q_t)du + \int_t^T B(c,T-u)\kappa\bar{v}du + \int_t^T \lambda_0\left(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\right)du \quad (2.49)$$

The integrals can be found analytically,

$$A(c,T-t) = c(T-t)(r_t - q_t) + \lambda_0(T-t)\Big(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\Big) \\ - a\kappa\bar{v}\int_t^T \frac{1-e^{-\gamma(T-u)}}{2\gamma - (\gamma+b)(1-e^{-\gamma(T-u)})}du \quad (2.50)$$

The integral in (2.50) can be found using Change of Variables method. Let us call $x = e^{-\gamma(T-u)}$

$$\int_t^T \frac{1-e^{-\gamma(T-u)}}{2\gamma - (\gamma+b)(1-e^{-\gamma(T-u)})}du = \int \frac{1-x}{2\gamma-(\gamma+b)(1-x)} \cdot \frac{dx}{\gamma x} \quad (2.51)$$

The right hand side of (2.51) can be separated further using method of partial fractions. Let us denote $D = \gamma + b$ and $E = 2\gamma$,

$$\int \frac{1-x}{2\gamma-(\gamma+b)(1-x)} \cdot \frac{dx}{\gamma x} \\ = \int \frac{dx}{E - Dx(1-x)} - \int \frac{dx}{E-D(1-x)} \\ = \frac{D}{D-E}\int \frac{dx}{E-D(1-x)} + \frac{1}{E-D}\int \frac{dx}{x} \\ = \frac{2\gamma}{(\gamma+b)(b-\gamma)}\Big[\ln(2\gamma) - \ln\big(2\gamma - (\gamma+b)(1-e^{-\gamma(T-t)})\big)\Big] \\ + \frac{\gamma(\gamma+b)(T-t)}{a\sigma_v^2} \quad (2.52)$$

One thing we should also consider before obtaining the result for $A(c,\tau)$ is that the discounting with r by the definition of $G_{a,b}$ in (2.17). Since r is a constant in our model, the integral $\exp\big(-\int_t^T r_u du\big)$ in the (2.17) gives us $\exp(-r\tau)$ as a constant. This constant can be added in the $A(c,\tau)$ after plugging (2.52) into (2.50). This compuation gives us the desired solution for $A(c,\tau)$;

$$A(c,\tau) = -r\tau + c\tau(r_t - q_t) + \lambda_0\tau\Big(\exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\Big) \\ - \frac{\kappa\bar{v}}{\sigma_v^2}\Big[(\gamma+b)\tau + 2\ln\big(1 - \frac{\gamma+b}{2\gamma}(1-e^{-\gamma\tau})\big)\Big] \quad (2.53)$$

Finally, we can combine the results obtained from (2.47) and (2.53) using (2.19) with (2.16), (2.17), (2.18). Therefore, in this section we have proved the following result:

Proposition 2.4.1. *(Call Price Formula) Let C_t be the time-t price of a European style call option with time to maturity τ, strike K and the underlying S_t follows the affine stochastic jump diffusion process as in* (2.2)*. Under risk-neutral measure $\mathbb{Q}$, C_t can be calculated with,*

$$C_t = G_{1,1}(-lnK) - KG_{0,1}(-lnK) \tag{2.54}$$

where,

$$G_{1,1}(-lnK; v, \tau, \vartheta) = \frac{\psi^{\vartheta}(1, v, \tau)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{Im[\psi^{\vartheta}(1 - iu, v, \tau)e^{iulnK}]}{u} du \tag{2.55}$$

$$G_{0,1}(-lnK; v, \tau, \vartheta) = \frac{\psi^{\vartheta}(0, v, \tau)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{Im[\psi^{\vartheta}(-iu, v, \tau)e^{iulnK}]}{u} du \tag{2.56}$$

as in [10] with,

$$\psi^{\vartheta}(c, v, \tau) = exp(A(c, \tau) + B(c, \tau)v + clnS_t) \tag{2.57}$$

$$B(c, \tau) = \frac{-a(1 - e^{-\gamma\tau})}{2\gamma - (\gamma + b)(1 - e^{-\gamma\tau})} \tag{2.58}$$

$$A(c, \tau) = -r\tau + c\tau(r_t - q_t) + \lambda_0\tau\left(exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*\right) \tag{2.59}$$

$$- \frac{\kappa\bar{v}}{\sigma_v^2}\left[(\gamma + b)\tau + 2ln\big(1 - \frac{\gamma + b}{2\gamma}(1 - e^{-\gamma\tau})\big)\right] \tag{2.60}$$

with $a = c(1 - c) - 2\lambda_1(exp(c\mu_j^* + \frac{1}{2}c^2\sigma_j^2) - 1 - c\mu^*)$, $b = c\sigma_v\rho - \kappa + \eta^v$, $\mu_j^* = ln(1 + \mu^*) - \sigma_j^2/2$ and $\gamma^2 = b^2 + a\sigma_v^2$.

CHAPTER 3

ESTIMATION ALGORITHM

Just as any other mathematical model, stochastic volatility models require parameters which need to be estimated to measure the option prices. There exists a lot of parameter estimation methods in the literature depending on the model specifications. Such examples about the commonly used estimation methods can be given as Maximum Likelihood Estimation (Friedman and Harris [12], Ait-Sahalia [1]) and Monte Carlo (Sandmann [19], Raggi [18]). In this thesis, we will focus on the Generalized Method of Moments to capture both stochastic volatility and jump dynamics without losing the analytical tractability.

In this section, we discuss how to estimate our parameters for the model. First, we give an introduction to Generalized Method of Moments, then we will introduce the implied-state GMM and moment generating function of the model as presented in Pan [16], and finally we will share the estimation results.

3.1 Generalized Method of Moments

Generalized Method of Moments (GMM) is a framework for deriving estimators by using assumptions about the moments of the random variables. These assumptions made on the sample moments provide population moment conditions. GMM calculates the sample moments by minimizing an objective function that derived from the assumptions about the moments. In this section, we will follow the Hamilton [14] and Alastair [13] to provide introductory material about GMM.

Let us start with denoting w_t as an $(h \times 1)$ vector of observed variables at time t, let θ denote $(a \times 1)$ vector of unknown coefficients with true value of θ_0, and let $h(\theta, w_t)$ be an $(r \times 1)$ vector-valued function with $h : (\mathbb{R}^a \times \mathbb{R}^h) \rightarrow \mathbb{R}^r$. True value of the model can be characterized as;

$$\mathbb{E}[h(\theta_0, w_t)] = 0 \tag{3.1}$$

This characterization of (3.1) is also known as the orthogonality condition. If we let $y_t = (w'_T, w'_{T-1}, \ldots, w'_1)$ to be a $(Th \times 1)$ vector that contains all observations in a sample size of T, and let $g(\theta; y_t)$ be $(r \times 1)$ vector valued function $g : \mathbb{R}^a \rightarrow \mathbb{R}^r$ that denotes the sample average of $h(\theta, w_t)$;

$$g(\theta; y_t) = \frac{1}{T} \sum_{t=1}^{T} h(\theta, w_t) \tag{3.2}$$

Now, we can apply the main idea behind the GMM; choosing θ so that the sample moment $g(\theta; y_t)$ gets as close as possible to population moment of zero. the GMM estimator $\hat{\theta}_T$ is the value of θ that minimizes

$$Q(\theta, y_T) = [g(\theta; y_T)]' W_T [g(\theta; y_T)] \tag{3.3}$$

where W_T is a sequence of $(r \times r)$ positive definite weighting matrix.

3.2 GMM Estimators and MGF

After having a brief introduction to GMM, we can now discuss the estimation strategy and Moment Generating Function (MGF) of the model which will help us to obtain moments as in Pan [16]. Let us start with fixing a time interval Δ which will be used to sample the continous-time process $\{S_t, V_t\}$ at discrete time $\{\Delta, 2\Delta, \ldots, N\Delta\}$. We will denote the discretized process as $\{S_n, V_n\}$. Time-n excess return can be calculated with

$$y_n = \log S_n - \log S_{n-1} - r\Delta \tag{3.4}$$

Let us suppose that, we can observe volatility V_n just as the return y_n. Since (3.4) is depends only on $\{S_n, V_n\}$, the problem will be as the usual GMM estimation.

Therefore, Pan [16] defines the n-th moment as in (3.1);

$$\mathbb{E}_{n-1}[h(y_{(n,n_y)}, V_{(n,n_v)}, \vartheta_{true})] = 0 \tag{3.5}$$

with ϑ_{true} being the vector of true model parameters, $h : \mathbb{R}^{n_y} \times \mathbb{R}^{n_v} \times \Theta \to \mathbb{R}^{n_h}$ is a vector valued function,

$$y_{(n,n_v)} = [y_n, y_{n-1}, \ldots, y_{n-n_y+1}]^\top \qquad V_{(n,n_v)} = [V_n, V_{n-1}, \ldots, V_{n-n_v+1}]^\top$$

are the n_y-history and n_v-history for some integers n_y and n_v respectively. Since the volatility V_n can not be observed directly, Pan [16] suggests to use the spot price S_n and option price C_n which can be observed in the market. The option pricing relation can be given as follows;

$$C_n = S_n f(V_n, \vartheta_{true}, r, q, \tau, k) \tag{3.6}$$

with constant interest rate r, dividend yield q, time to maturity τ and strike to spot ratio $k = K/S_n$. Pan suggests that rather than true model parameters ϑ_{true}, using any other set of parameters $\vartheta \in \Theta$, one can still obtain unobserved volatility V_n from the implied volatility V_n^ϑ by solving

$$C_n = S_n f(V_n^\vartheta, \vartheta, r, q, \tau, k) \tag{3.7}$$

Pan states that backing out volatility from options is not a new concept since the famous Black-Scholes, however this specification differs from the Black-Scholes formula by being the option implied volatility V_n^ϑ parameter dependent. He mentions that this parameter dependency criterion make the difference from the usual GMM estimation and hence adding the new term Implied-State GMM. As in the GMM specification, the Implied State GMM is also satisfies the consistency and the asymptotic normality properties. For a detailed proof about the IS-GMM specification, we refer [16, Appendix C].

After obtaining option implied volatility V_n^ϑ, one can now apply (3.2) to $\{y_n, V_n\}$ as follows;

$$G_N(\vartheta) = \frac{1}{N}\sum_{n=1}^{N} h(y_{(n,n_y)}, V_{(n,n_v)}^\vartheta, \vartheta) \tag{3.8}$$

After finding $G_N(\vartheta)$, we can define the "implied-state" GMM estimator as (3.3),

$$\hat{\vartheta}_N = \underset{\vartheta \in \Theta}{\text{argmin}} G_N(\vartheta)^\top \mathcal{W} G_N(\vartheta) \tag{3.9}$$

where $\mathcal{W}$ is a weighting matrix with the i^{th} component is $1/\sqrt{\text{var}(\epsilon_i)}$.

We can now be able to define the moment conditions got our estimator. As Pan suggests, one can find optimal moment conditions by using the moment generating function of (y, V) for any u_y and u_v in $\mathbb{R}$ as follows;

$$\mathbb{E}_n\big[\text{exp}(u_y y_{n+1} + u_v V_{n+1})\big] = \phi(u_y, u_v, V_n) \tag{3.10}$$

In our work, we will use the moment generating function as he presented in Pan [16, Appendix D].

Proposition 3.2.1. *(Moment Generating Function) Under certain integrability conditions as presented in Duffie [10], the date-n conditional moment generating function of (y_{n+1}, V_{n+1}) can be defined as,*

$$\phi(u_y, u_v, v) = exp(A(u_y, u_v) + B(u_y, u_v)v) \tag{3.11}$$

with the coefficients A and B defined as,

$$B(u_y, u_v) = -\frac{a(1 - exp(-\gamma\Delta)) - u_v[2\gamma - (\gamma - b)(1 - exp(-\gamma\Delta))]}{2\gamma - (\gamma + b)(1 - exp(-\gamma\Delta)) - u_v\sigma_v^2(1 - exp(-\gamma\Delta))} \tag{3.12}$$

$$A(u_y, u_v) = -\frac{\kappa_v\bar{v}}{\sigma_v^2}\left((\gamma + b)\delta + 2ln\Big[1 - \frac{\gamma + b + \sigma_v^2 u_v}{2\gamma}(1 - e^{-\gamma\Delta})\Big]\right) \tag{3.13}$$

$$+ \left(exp\Big(u_y\mu_j + \frac{u_y^2\sigma_j^2}{2}\Big) - 1 - u_y\mu^*\right)\lambda_0\Delta \tag{3.14}$$

with $b = \sigma_v\rho u_y - \kappa_v$, $a = -u_y^2 - 2u_y[\eta_s - 1/2 - \lambda_1\mu^*] - 2\lambda_1(exp(u_y\mu_j + u_y^2\sigma_j^2/2) - 1)$, and $\gamma = \sqrt{b^2 + a\sigma_v^2}$

We can now find the joint conditional moments of (y, V) by taking the derivatives of (3.11),

$$\mathbb{E}_n(y_{n+1}^i, V_{n+1}^j) = \frac{\partial^{(i+j)}\phi(u_y, u_v, V_n)}{\partial^i u_y \partial^j u_v}\bigg|_{u_y=0, u_v=0} \qquad i, j \in \{0, 1, \dots\} \tag{3.15}$$

Pan provides a scheme for calculating higher orders of i and j in his work. We refer the reader to Pan [16, Appendix D] for the detailed version of the scheme.

Now let us define the simple moment conditions using (3.5), as in [16];

$$\epsilon_n^{y1} = y_n - \mathbb{E}[y_n|\mathscr{F}_{n-1}], \quad \epsilon_n^{y2} = y_n^2 - \mathbb{E}[y_n^2|\mathscr{F}_{n-1}]$$
$$\epsilon_n^{y3} = y_n^3 - \mathbb{E}[y_n^3|\mathscr{F}_{n-1}], \quad \epsilon_n^{y4} = y_n^4 - \mathbb{E}[y_n^4|\mathscr{F}_{n-1}]$$
$$\epsilon_n^{v1} = V_n - \mathbb{E}[V_n|\mathscr{F}_{n-1}], \quad \epsilon_n^{v2} = V_n^2 - \mathbb{E}[V_n^2|\mathscr{F}_{n-1}]]$$
$$\epsilon_n^{yv} = y_n V_n - \mathbb{E}[y_n V_n|\mathscr{F}_{n-1}]]$$

with

$$\mathbb{E}(\epsilon_n) = 0, \quad \epsilon_n = [\epsilon_n^{y1}, \epsilon_n^{y2}, \epsilon_n^{y3}, \epsilon_n^{y4}, \epsilon_n^{v1}, \epsilon_n^{v2}, \epsilon_n^{yv}]^\top \tag{3.16}$$

In his work, Pan mentions that the moment conditions in (3.16) provides natural and testable results on lower degree moments; however, he also introduces a method that allows him to compute much more efficient moment conditions. In our work, we do not implement this additional step presented by Pan and we advice the reader to refer [16, Section 3.2] of the related work. In the next section, we first run the estimation algorithm using the data provided by Pan [16] to see the simplified approach's performance; later on we will attempt to apply the same scheme to our TUPRS data.

CHAPTER 4

NUMERICAL EXAMPLES AND ESTIMATION RESULTS

In this chapter we present some numerical examples on the performance of the Generalized Method of Moments (GMM) estimation procedure outlined in the previous chapter. Recall that this is a simplified version of the estimation procedure used in [16], which can be summarized as follows: the process y is defined from S using:

$$y_n = \log(S_{\Delta n}) - \log(S_{\Delta(n-1)}) - r\Delta.$$

In actual data the V process is not directly observable and we have the option price C_t as observable; to overcome this [16] defines

$$V_t^{\vartheta} = g(C_t, \vartheta)$$

where g is the inverse of the option pricing function with respect to the volatility variable (see [16, page 40]) assuming that the model parameters equal ϑ. Then GMM is applied to the sequence (y, V^{ϑ}) using the following seven moment conditions:

$$\mathbb{E}[\varepsilon_i] = 0, i \in \{1, 2, 3, 4, 5, 6, 7\}, \tag{4.1}$$

where

$$\begin{aligned}
&\varepsilon_1 = y_n - \mathbb{E}[y_n|\mathscr{F}_{n-1}], \varepsilon_2 = y_n^2 - \mathbb{E}[y_n^2|\mathscr{F}_{n-1}],\\
&\varepsilon_3 = y_n^3 - \mathbb{E}[y_n^3|\mathscr{F}_{n-1}], \varepsilon_4 = y_n^4 - \mathbb{E}[y_n^4|\mathscr{F}_{n-1}] = 0,\\
&\varepsilon_5 = V_n - \mathbb{E}[V_n|\mathscr{F}_{n-1}]] = 0, \varepsilon_6 = V_n^2 - \mathbb{E}[V_n^2|\mathscr{F}_{n-1}] = 0,\\
&\varepsilon_7 = y_n V_n - \mathbb{E}[y_n V_n|\mathscr{F}_{n-1}] = 0.
\end{aligned}$$

In the implementation we use the explicit formulas derived for these moments in [16, Appendix D]. In each step of the GMM, V^{ϑ} must be recomputed; in these computations one takes ϑ equal to the parameter estimates given by the last iteration of GMM.

As already noted this is a simplified version of the estimation procedure used in [16]; the procedure in [16] contains an additional step in which ε are further processed to obtain optimal moment conditions. We skip this step to simplify our first implementation of this complex estimation algorithm.

Let $G_N(\vartheta)$ denote the sample analog of the moment conditions; then GMM estimates ϑ by minimizing

$$\mathscr{E}(\vartheta) = G_N(\vartheta)'\mathscr{W}G_N(\vartheta),$$

where A' denotes the transpose of A. Once again, to ease computations, we use as $\mathscr{W}$ a diagonal matrix where the i^{th} component is $1/\sqrt{\text{var}(\epsilon_i)}$.

Before we apply the estimation algorithm to data we wanted to see its performance on data obtained from simulations. In the next section we report our simulation results.

4.1 Simulation Results

To see how the GMM algorithm performs in simulations we proceeded as follows. We fixed the parameter values to ϑ^* and simulated (y, V) for these parameter values; the compute the simulated paths we discretized the continuous time dynamics using an Euler scheme. The call price process C is then computed from (y, V) using the option price formula available for the model. We then reestimate ϑ^* from the simulated data (y, C) and compare the estimation to ϑ^*.

For ϑ^* we use the parameter values listed in [16, Table 3, page 25]; these are the parameter values estimated for the model in [16] when the algorithm is applied to (y, C) arising from S&P 500 observed from January 1989 to December 1996.

To run the GMM algorithm an initial guess ϑ_0 must be provided; for this initial guess we take a random perturbation of ϑ^* (each component of ϑ^* is scaled by a random number between 0 and 1). The ϑ^* and ϑ_0 used in our simulations are listed in Table 4.1.

In addition to these parameters we take $\Delta = 1./50$, the maturity of the option to be $T = 1.2$ and $r = 0.1$; the simulation is run for $N = 50$ steps. The sample paths of y, V and C are shown in Figures 4.1 and 4.2.

Table 4.1: ϑ^* and ϑ_0 used in the simulations.

	$\bar{\nu}$	σ_v	κ_v	η_v	ρ	λ	η_s	μ	μ^*	σ_J
ϑ^*	0.153	0.3	0.3	3.6	-0.53	12.3	3.6	-0.8/100	-19.2/100	3.87/100
ϑ_0	0.02	0.191	0.236	2.29	-0.463	4.26	0.179	-0.33/100	-14.57/100	3.34/100.

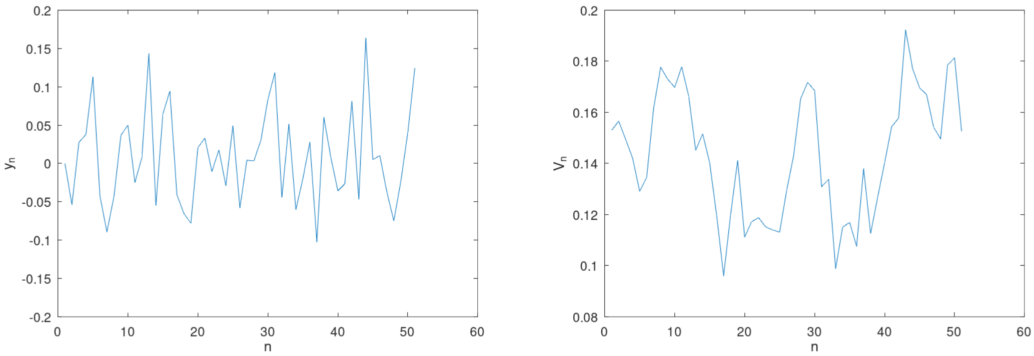

Figure 4.1: Simulated paths of y and V

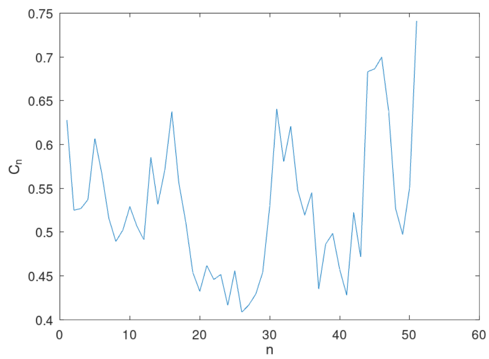

Figure 4.2: Simulated path of the call price

In our runs of the GMM algorithm we observed that it typically converges after 3 iterations. The GMM estimate ϑ_4 of ϑ^* with initial guess ϑ_0 after 4 iterations is listed in Table 4.2; for ease of reference ϑ^* and ϑ_0 are also relisted in the same table.

Table 4.2: GMM estimates for the simulated paths

	$\bar{\nu}$	σ_v	κ_v	η_v	ρ	λ	η_s	μ	μ^*	σ_J
ϑ^*	0.153	0.3	0.3	3.6	-0.53	12.3	3.6	-0.8/100	-19.2/100	3.87/100
ϑ_0	0.02	0.191	0.236	2.29	-0.463	4.26	0.179	-0.33/100	-14.57/100	3.34/100
ϑ_4	0.0085	0.168	0.251	2.29	-0.463	4.26	0.199	-0.31/100	-22.9/100	0.1/100

The GMM error for ϑ^*, ϑ_0 and ϑ_4 are $\mathscr{E}(\vartheta^*) = 1.87 \times 10^{-3}$, $\mathscr{E}(\vartheta_0) = 3.88 \times 10^{-3}$ and $\mathscr{E}(\vartheta_4) = 3.4 \times 10^{-3}$.

The work [16, page 15] makes the following comments on the moment conditions (4.1):

> This choice of moment conditions is intuitive and provides some natural and testable conditions on lower moments and cross moments of y and V. But these are not the most efficient moment conditions.

We think that the above results further support this claim. In particular, the above simulation results suggest the following: the GMM algorithm based on the moment conditions (4.1) is not very sensitive to the actual model parameters. The algorithm seems to converge quickly to a point around the initial guess ϑ_0. When we repeat the above simulation study for different parameter values we get similar results. Based on these observations and our implementation, we find that the GMM algorithm based on the standard moment conditions (4.1) is not very useful in identifying the actual model parameters.

4.2 Application to Data

Given the poor simulation performance presented in the previous section of GMM based on the moment conditions (4.1) we don't think that it can give reliable results when applied to real data. Nonetheless, we did apply it to see how it runs on real data and whether it behaves in a way similar to the way it did in simulations.

For the data we use the stock price of Turkiye Petrol Rafinerileri AS (TUPRS) and a call option written on this asset. The time interval is the trading days between February 1 and March 15 of 2024; we use daily price data. The Feb 1 price of the stock is $S_0 = 154.7$; in the given time interval the stock traded between 154.3 and 173. For the call option we use the one with strike $K = 170$ and maturity equal to April 30, 2024. We take the interest rate in this time interval to be $r = 0.45$, which is the policy rate of the Turkish Central Bank in this time interval. TUPRS paid no dividends in this time interval, so we take $q = 0$. Finally we take $\Delta = 1/260$. The graphs of y and C for the data are shown in Figure 4.3.

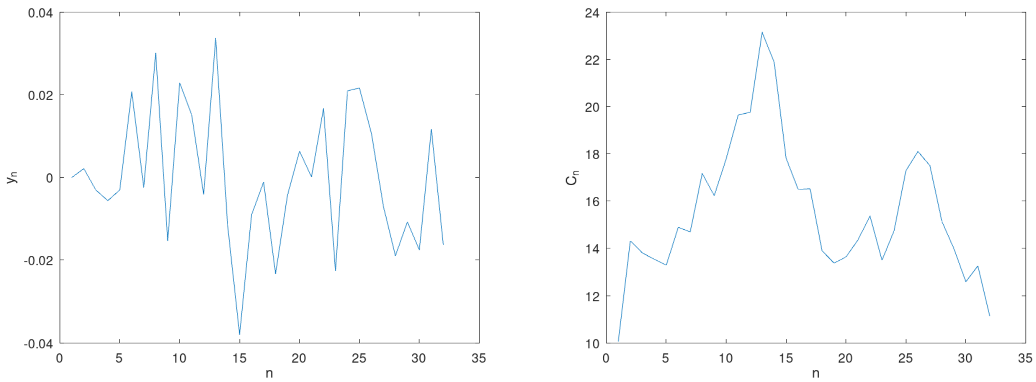

Figure 4.3: Paths of y and C for the data

The results of the GMM estimation using the moment conditions (4.1) for this data are given in Table 4.3. As in the simulations, GMM converges after 3 or 4 iterations. For the initial parameter values ϑ_0 supplied to the GMM algorithm we use one that was used in the simulations above.

Table 4.3: GMM estimates for the data

	$\bar{\nu}$	σ_v	κ_v	η_v	ρ	λ	η_s	μ	μ^*	σ_J
ϑ_0	0.02	0.191	0.236	2.29	-0.46341	4.2638	0.179	-0.33/100	-14.57/100	3.34/100
ϑ_4	0.03	0.203	0.229	2.29	-0.46373	4.2642	0.1824	0.71/100	-15.85/100	0.1/100.

Qualitatively we observe a behavior similar to the one we observed in the simulation results. Namely, the GMM algorithm based on (4.1) converges in several steps to a point near the initial parameter estimate ϑ_0. We repeated the estimation procedure with a number of different initial guesses ϑ_0 and observed similar results.

CHAPTER 5

CONCLUSION

In this thesis, we studied the seminal work of Pan [16] on the estimation of the jump risk premia in a stochastic volatility models with jumps. We gave a detailed derivation of the characteristic functions used in this work. We then simulated the model and applied the IS-GMM algorithm of [16] to the simulated data using ordinary moment conditions. As discussed in the previous section the use of ordinary moment conditions seem to lead to poor estimation results. We observed a similar performance when we applied the algorithm to actual price data. A natural next step is to repeat this study with the optimized moment conditions used in [16]. Another possible future work is to allow jumps in the volatility process as is done in [9].

REFERENCES

[1] Y. Ait-Sahalia and R. Kimmel, Maximum likelihood estimation of stochastic volatility models, Journal of Financial Economics, 83, pp. 413–452, February 2007.

[2] T. G. Andersen and B. E. Sørensen, Gmm estimation of a stochastic volatility model: A monte carlo study, Journal of Business and Economic Statistics, 14(3), pp. 328–352, 1996, ISSN 07350015.

[3] G. Bakshi, C. Cao, and Z. Chen, Empirical performance of alternative option pricing models, The Journal of Finance, 52(5), pp. 2003–2049, 1997.

[4] L. V. Ballestra and L. Cecere, A fast numerical method to price american options under the bates model, Computers and Mathematics with Applications, 72(5), pp. 1305–1319, 2016, ISSN 0898-1221.

[5] D. S. Bates, Post-'87 crash fears in the s&p 500 futures option market, Journal of Econometrics, 94(1), pp. 181–238, 2000, ISSN 0304-4076.

[6] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, 81(3), pp. 637–654, 1973, ISSN 00223808.

[7] A. E. Bolko, K. Christensen, M. S. Pakkanen, and B. Veliyev, A gmm approach to estimate the roughness of stochastic volatility, Journal of Econometrics, 235(2), pp. 745–778, 2023, ISSN 0304-4076.

[8] M. Chernov and E. Ghysels, A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation, Journal of Financial Economics, 56(3), pp. 407–458, 2000.

[9] G. Deng, Option pricing under two-factor stochastic volatility jump-diffusion model, Complexity, 2020, p. 1–15, September 2020.

[10] D. Duffie, J. Pan, and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, Econometrica, 68(6), pp. 1343–1376, November 1999.

[11] D. Duffie and K. J. Singleton, Simulated moments estimation of markov models of asset prices, Econometrica, 61(4), pp. 929–952, 1993, ISSN 00129682, 14680262.

[12] M. Friedman and L. Harris, A maximum likelihood approach for non-gaussian stochastic volatility models, Journal of Business and Economic Statistics, 16(3), pp. 284–91, 1998.

[13] A. Hall, *Generalized Method of Moments*, pp. 230–255, Oxford University Press, November 2007, ISBN 9780470996249.

[14] J. D. Hamilton, *Time Series Analysis*, Princeton University Press, 1994.

[15] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, Review of Financial Studies, 6(2), p. 327–343, 1993.

[16] J. Pan, The jump-risk premia implicit in options: Evidence from an integrated time-series study, Journal of Financial Economics, 63(1), p. 3–50, 2002.

[17] P. Protter, *Stochastic integration and differential equations*, Springer, 2005.

[18] D. Raggi and S. Bordignon, Sequential monte carlo methods for stochastic volatility models with jumps, Department of Economics, University of Bologna, January 2006.

[19] G. Sandmann and S. J. Koopman, Estimation of stochastic volatility models via monte carlo maximum likelihood, Journal of Econometrics, 87(2), pp. 271–301, 1998, ISSN 0304-4076.

[20] S. E. Shreve, *Stochastic calculus for finance II: Continuous-time models*, Springer, 2010.

[21] J. Toivanen, *A Componentwise Splitting Method for Pricing American Options Under the Bates Model*, pp. 213–227, Springer Netherlands, 2010, ISBN 978-90-481-3239-3.